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ROBUST ANALYSIS AND GLOBAL OPTIMIZATION

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1. INTRODUCTION

Let X be a Hausdorff topological space, S be a compact subset of X and f be a lower semicontinuous (l.s.c.) function defined on S. Then the minimum of f over S exists:

$$\bar{c} = \min_{x \in S} f(x) \tag{1.1}$$

and the set of minima

$$\bar{H} = \{x | f(x) = \bar{c},\ x \in S\} \tag{1.2}$$

is nonempty.

Up till now people rarely consider the problem of finding solution of (1.1) under such loose conditions not because it is useless. The objective function may be discontinuous; the constrained set may be disconnected. But many problems from natural and social sciences, as well as from industrial applications do require minimizing a discontinuous function. On the other hand, problems in applications may require that the constrained set S be disconnected, by the reason of physical forbiddance. If we only considered the problem of finding a local minimum, it would not bother us whether S is connected or not.

But we still have to put certain restrictions on the set S and on the function f. This is why the robust analysis should be studied.

In this work we extended the earlier works [1] and [2]. We first study the properties of robust sets and robust functions. And then the minimization problem of a robust function over a robust compact set is considered by the integral approach. Optimality conditions in [3] and the algorithm in [4] and [5] are extended to this case. Numerical tests and industrial applications [6] show that the algorithm is effective.

2. ROBUST SETS AND ROBUST POINTS

We begin with definitions of a robust set and a robust point. Let X be a topological space and D be a subset of X.

DEFINITION 2.1. *A set D is said to be robust iff*

$$cl\ D = cl\,(int\ D). \tag{2.1}$$

DEFINITION 2.2. *A point $x \in cl\ D$ is said to be robust to the set D iff for each neighborhood*

$$N(x) \cap int\ D \neq \emptyset. \tag{2.2}$$

An open set is robust. The concept of robustness is a generalization of that of openness. A closed set may be nonrobust. For instance, a point x is closed in $\mathbf{R}^1$ but it is nonrobust. Note

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that the concept of robustness is closely related to the given topology. A subset of integers is not robust on $\mathbf{R}^1$, but it is robust with respect to discrete topology on the set of integers.

REMARK. *We only define the robustness of the points in cl D because if $x \notin$ cl D, then there is a neighborhood $N(x)$ of x such that $N(x) \cap$ cl $D = \emptyset$, i.e., (2.2) does not hold. Thus, the points which are not contained in cl D are always nonrobust.*

The following theorem shows that each point of a robust set is robust to this set, and vice versa.

THEOREM 2.1. *A set D is robust if and only if each point of D is robust to D.*

PROOF. Suppose there is a point $x \in D$ which is nonrobust to D, then there exists a neighborhood $N(x)$ of x such that $N(x) \cap \text{int } D = \emptyset$. It implies $x \notin$ cl (int D). Thus, $x \in$ cl $D \backslash$cl (int D) $\neq \emptyset$. D is then nonrobust.

Conversely, suppose each point of D is robust to D but, by contrary, D is nonrobust. That is, $A =$ cl $D\backslash$cl (int D) $\neq \emptyset$. Take a point $x \in A$. Since $x \notin$ cl (int D), we can find a neighborhood $N(x)$ of x such that $N(x) \cap \text{int } D = \emptyset$. Since $x \in$ cl D, $N(x) \cap D \neq \emptyset$. Take a point $x_1 \in N(x) \cap D$ and take a neighborhood $N_1(x_1)$ of x_1 such that $N_1(x_1) \subset N(x)$. Then $N_1(x_1) \cap \text{int } D = \emptyset$. This means that x_1 is nonrobust to D. We have a contradiction. ∎

Suppose a point x is robust to both D and G, then the point x may be nonrobust to their intersection $D \cap G$. For instance, let $D = [0,1]$ and $G = [0,1/2] \cup [1,2]$. They are robust on $\mathbf{R}^1$. The point $x = 1 \in D \cap G = [0,1/2] \cup \{1\}$ is nonrobust to $D \cap G$.

THEOREM 2.2. *Suppose x is robust to D and $x \in$ int G, then x is robust to $D \cap$ int G and also to $D \cap G$.*

PROOF. For each neighborhood $N_1(x) \subset N(x) \cap$ int G, we have $N_1(x) \cap$ int $D \neq \emptyset$ because x is robust to D. Now we have

$$N(x) \cap \text{ int } (D \cap \text{int } G) = N(x) \cap \text{ int } G \cap \text{ int } D \supset N_1(x) \cap \text{ int } D \neq \emptyset.$$

Therefore, x is robust to $D \cap$ int G and, thus, to $D \cap G$ (which may be a larger set). ∎

With the help of Theorem 2.1 and Theorem 2.2 we can easily prove the following theorems.

THEOREM 2.3. *The union of robust sets is robust.*

THEOREM 2.4. *The intersection of a robust set and an open set is robust.*

The following statements hold, and can be proved, using the above theorems:

(1) if D is robust then cl D is also robust;
(2) a point x is robust to D if and only if $x \in$ cl (int D);
(3) if D is robust and F is closed, then $D\backslash F$ is robust;
(4) a set D is robust if and only if $\partial D = \partial \text{int } D$, where $\partial D =$ cl $D \backslash$int D, the boundary of D.

3. ROBUST FUNCTIONS

Let $f : X \to \mathbf{R}$ be a real valued function defined on a topological space X. In this section we will consider a class of discontinuous functions related to the concepts of robust sets and points. A function f is said to be upper semi-continuous (u.s.c.) iff the set

$$F_c = \{x | f(x) < c\} \tag{3.1}$$

is open for each real number c. A function f is said to be u.s.c. at a point x_0 iff $x_0 \in F_c$ implies $x_0 \in$ int (F_c). We generalize these concepts to robust functions.

DEFINITION 3.1. *A function f is said to be robust iff the set $F_c = \{x | f(x) < c\}$ is robust for each real number c.*

Definition 3.2. *A function f is said to be robust at a point x_0 iff $x_0 \in F_c$ implies x_0 is robust to F_c.*

An u.s.c. function is robust, so is a continuous function. A monotone function f on $\mathbf{R}^1$ is robust. Indeed, suppose f is increasing, then $F_c = (-\infty, \alpha)$, where $c = f(\alpha)$, the point α may be contained in F_c or not. In both cases F_c is robust. If f is u.s.c. at a point x_0, then f is also robust at the point x_0.

The following theorem is expected.

Theorem 3.1. *A function is robust if and only if it is robust at each point.*

The sum of two robust functions may be nonrobust. For example, let

$$f_1(x) = \begin{cases} 1, & x < 0 \\ 0, & x \geq 0, \end{cases} \quad \text{and} \quad f_2(x) = \begin{cases} 0, & x \leq 0 \\ 1, & x > 0, \end{cases} \tag{3.2}$$

then f_1 and f_2 are robust. But the sum of them

$$f(x) = f_1(x) + f_2(x) = \begin{cases} 1, & x \neq 0, \\ 0, & x = 0, \end{cases} \tag{3.3}$$

is nonrobust at $x = 0$.

Theorem 3.2. *Suppose that f is robust at x_0 and g is u.s.c. at x_0 (for division g is supposed to be l.s.c. at x_0). Then the following functions are robust at x_0.*

(1) $\alpha f \quad (\alpha \geq 0)$;
(2) $f + g$;
(3) $f \cdot g \quad (g(x_0) > 0)$;
(4) $f/g \quad (g(x_0) > 0)$.

It is easy to prove the following proposition from Theorem 3.1.

Proposition 3.3. *Suppose that f is robust and g is u.s.c. (for division g is supposed to be l.s.c.), then the following functions are robust: (1) αf ($\alpha \geq 0$); (2) $f + g$; (3) $f \cdot g$ ($g(x) > 0$) and (4) f/g ($g(x) > 0$).*

If $f_\alpha(x)$ is robust at a point x_0 for each $\alpha \in \Lambda$, then the function $f(x) = \inf_{\alpha \in \Lambda} f_\alpha(x)$ is also robust at the point x_0. The limit of a decreasing sequence of robust functions preserves the robustness, and so on. We would not discuss the properties and structure of robust functions in detail which are beyond the scope of the paper. Before transferring to global optimization, we would like to mention a property related to the epigraph of a function. Recall that an epigraph of a function f is defined as (see [7]):

$$\text{epi }(f) = \{(x, c) | f(x) \leq c\}. \tag{3.4}$$

The epigraph is a subset of the product space $X \times \mathbf{R}$.

Theorem 3.3. *A function f is robust at x_0 if and only if each point $(x, c) \in$ epi (f) is robust to the set epi (f) in the product space $X \times \mathbf{R}$ with the product topology.*

Therefore, a function f is robust if and only if its epigraph epi (f) is robust in the product space $X \times \mathbf{R}$.

4. RELATIVE ROBUSTNESS

Consider the following minimization problem:

$$\bar{c} = \min_{x \in S} f(x), \tag{4.1}$$

where we assume that

(A1) S is a compact set in X;
(A2) $f : S \to \mathbf{R}$ is a l.s.c. function.

To study such a constrained problem the concept of relative robustness should be investigated.

DEFINITION 4.1. *An objective function f is said to be relatively robust to S at a point $x_0 (\in cl\, S)$ if $x_0 \in F_c = \{x | f(x) < c\}$ implies x_0 is robust to $F_c \cap S$.*

The following proposition gives us sufficient conditions of relative robustness of a function.

PROPOSITION 4.1. *If (1) f is robust at x_0 and $x_0 \in int\, S$; or (2) x_0 is robust to S and f is u.s.c. at x_0, then f is relatively robust to S at x_0.*

These conditions are sufficient. For instance, let $S = [-A, 0] \cup [1, A]$, where $A > 1$,

$$f_1(x) = \begin{cases} 1 - x, & x \in [-A, 0) \\ 1, & x = 0 \\ x, & x \in (0, A]. \end{cases} \tag{4.2}$$

and

$$f_2(x) = \begin{cases} 1 - x & x \in [-A, 0) \\ -1, & x = 0 \\ x, & x \in (0, A]. \end{cases} \tag{4.3}$$

In both cases, $x = 0$ is neither in int S, nor f is continuous at $x = 0$. However, f_1 is relatively robust to S at $x = 0$, and f_2 is not.

A concept of inf-robustness is introduced in [8] for minimization problems.

DEFINITION 4.2. *A set is said to be inf-robust with respect to minimization problem (4.1) iff for each $c_0 > \bar{c}$ there is $c (\leq c_0)$ such that $F_c \cap S$ is a nonempty robust set.*

In [8] we only consider the case of continuous objective functions, and the definition of inf-robustness is simplified as follows: a set S is inf-robust iff there is a real number c such that $F_c \cap S$ is nonempty and robust.

If S is inf-robust, then f is relatively robust to S at each (global) minimum point. But for some problems the set of global minima may be empty. In this case we can utilize the concept of inf-robustness to construct an algorithm to find the infimum of f over S. For example, let

$$S = \langle 0, 1], \quad \text{notation } \langle \text{ meaning either (or [;}$$

$$A = \bigcup_{k=1}^{\infty} (1/2k, 1/(2k-1)], \qquad B = S \backslash A = \bigcup_{k=1}^{\infty} (1/2k, 1/(2k-1)];$$

$$f(x) = \begin{cases} x, & x \in A, \\ x, & x \in B \cap \text{ (irrational numbers)}, \\ 1, & x \in B \cap \text{ (rational numbers)}, \end{cases} \tag{4.4}$$

Then f is inf-robust, but the set of minimum points is empty. The function f is relatively robust to S at $x = 0$, which is in the closure of S.

In the following considerations we assume:

(R) f is relatively robust to S at a global minimum point of (4.1).

As an example, consider the following combinatorial optimization problem.
Let $Z_+^n = \{z = (z_1, \ldots, z_n) |$ where z_i is a positive integer, $i = 1, \ldots, n\}$, S be a finite subset of Z_+^n, $f : S \to \mathbf{R}$ be a function defined on S, $f(z) = f(z_1, \ldots, z_n)$. The problem is to find the minimum value

$$\bar{c} = \min_{z \in S} f(z) \tag{4.5}$$

and the set of minima,

$$\bar{H} = \{z \in S | f(z) = \bar{c}\}. \tag{4.6}$$

For this case $\bar{H}$ is nonempty.

We define

$$D = \{x = (x_1, \ldots, x_n) \in \mathbf{R}^n | ([x_1 + 1/2], \ldots, [x_n + 1/2]) \in S\} \tag{4.7}$$

and

$$F(x) = f([x_1 + 1/2], \ldots, [x_n + 1/2]), \tag{4.8}$$

where $[\alpha]$ denotes the integer part of the real number α. D is a union of cubes, they are robust in $\mathbf{R}^n$. For each real number c, the set $\{x | F(x) < c\}$ is also a union of cubes (or empty). Thus, D is a robust set in $\mathbf{R}^n$ and F is a robust function. Let $\bar{x}$ be a global minimum point of F over D, i.e.,

$$F(\bar{x}) = \min_{x \in D} F(x), \tag{4.9}$$

then $\bar{x} \in \operatorname{int} D$ (or one can find a point x_1, in the same cube with x such that $x_1 \in \operatorname{int} D$). Therefore, the assumption (R) is satisfied.

5. OPTIMALITY CONDITIONS

In order to find global minima with the integral approach, a special class of measure spaces is required. Let X be a normal topological space, Ω be a σ-field of subsets of X. A measure space (x, Ω, μ) is said to be a Q-measure space if

(M1) each open set is in Ω;
(M2) the measure of each nonempty open set is positive;
(M3) the measure of each compact set is bounded.

The Lesbegue measure in $\mathbf{R}^n$ is a Q-measure; a non degenerate Gaussian measure on a separable Hilbert space is also a Q-measure. A specific measure space can be utilized to solve a specific minimization problem.

The following lemma is a sufficient condition for global optimality.

LEMMA 5.1. *Suppose the assumptions (A1), (A2), (R), (M1) and (M2) hold, and* $S \cap H_c \neq \emptyset$, *where* $H_c = \{x | f(x) \leq c\}$ *is a level set of the function* f. *If*

$$\mu(S \cap H_c) = 0, \tag{5.1}$$

then c *is the global minimum value of* f *over* S *and* $S \cap H_c$ *is the set of global minima.*

PROOF. Suppose, on the contrary, that c is not the global minimum value and $\bar{c} < c$ is. Let $2\eta = c - \bar{c} > 0$. There is a global minimiser x such that $\bar{c} = f(x)$ and f is relatively robust to S at x because of assumption (R). We have $x \in F_{c-\eta}$. Thus, $N(x) \cap \operatorname{int} (S \cap F_{c-\eta}) \neq \emptyset$, where $N(x)$ is a neighborhood of x. We now have $\operatorname{int} (S \cap F_{c-\eta}) \neq \emptyset$ and

$$\operatorname{int} (S \cap F_{c-\eta}) \subset S \cap H_c, \tag{5.2}$$

which implies with the assumption (M2) that

$$\mu(S \cap H_c) \geq \mu(\operatorname{int} (S \cap F_{c-\eta})) > 0, \tag{5.3}$$

which is a contradiction. ∎

The condition (5.1) is a sufficient one. If c is the global minimum value of f over S, it may happen that $\mu(S \cap H_c) > 0$. From Lemma 5.1, if $c > \bar{c} = \min_{x \in S} f(x)$, then $\mu(S \cap H_c) > 0$.

We now proceed to define the concepts of mean value, variance and higher moments of f over its level set and constrained set S as in [3]. These concepts are closely related to optimality conditions and to the algorithm for finding global minima.

DEFINITION 5.1. *Let* $c > \bar{c} = \min_{x \in S} f(x)$ *and suppose that assumptions (A1), (A2), (R), (M1), (M2) and (M3) hold. We define the mean value, variance, modified variance and* m*-th moment* *(centered at a) of a function f over its level set and the constrained set S, respectively, as follows:*

$$M(f,c;S) = \frac{1}{\mu(S \cap H_c)} \int_{S \cap H_c} f(x)\, d\mu, \tag{5.4}$$

$$V(f,c;S) = \frac{1}{\mu(S \cap H_c)} \int_{S \cap H_c} (f(x) - M(f,c;S))^2\, d\mu \tag{5.5}$$

$$V_1(f,c:S) = \frac{1}{\mu(S \cap H_c)} \int_{S \cap H_c} (f(x) - c)^2\, d\mu \tag{5.6}$$

and

$$M_m(f,c;a;S) = \frac{1}{\mu(S \cap H_c)} \int_{S \cap H_c} (f(x) - a)^m\, d\mu \qquad m = 1,2,\ldots \tag{5.7}$$

Since the function f is measurable, $H_c \cap S$ is compact and $\mu(S \cap H_c) > 0$, so (5.4)–(5.7) are well defined. When $c = \bar{c}$, $\mu(S \cap H_c)$ may be equal to zero. Definition 5.1 has to be extended by a limit process.

DEFINITION 5.2. *Under the assumptions of Definition 5.1, we can extend it to $c \geq \bar{c}$ as follows:*

$$M(f,c;S) = \lim_{c_k \downarrow c} \frac{1}{\mu(S \cap H_{c_k})} \int_{S \cap H_{c_k}} f(x)\, d\mu, \tag{5.8}$$

$$V(f,c;S) = \lim_{c_k \downarrow c} \frac{1}{\mu(S \cap H_{c_k})} \int_{S \cap H_{c_k}} (f(x) - M(f,c;S))^2\, d\mu, \tag{5.9}$$

$$V_1(f,c;S) = \lim_{c_k \downarrow c} \frac{1}{\mu(S \cap H_{c_k})} \int_{S \cap H_{c_k}} (f(x) - c)^2\, d\mu, \tag{5.10}$$

and

$$M_m(f,c;a;S) = \lim_{c_k \downarrow c} \frac{1}{\mu(S \cap H_{c_k})} \int_{S \cap H_{c_k}} (f(x) - a)^m\, d\mu, \qquad m = 1,2,\ldots \tag{5.11}$$

The limits exist and they are independent of choices of $\{c_k\}$. The extended concepts are well defined and consistent with those of Definition 5.1. The proofs are similar to those in [3]. With these concepts we characterize the global optimality as follows.

THEOREM 5.1. *Under the assumptions (A1), (A2), (R), (M1), (M2) and (M3), the following statements are equivalent:*

(1) *$\bar{x} \in S$ is the global minimum point of (4.1) and $\bar{c} = f(\bar{x})$ is the global minimum value;*
(2) *$M(f,\bar{c};S) = \bar{c}$;*
(3) *$V(f,\bar{c};S) = 0$;*
(4) *$V_1(f,\bar{c};S) = 0$;*
(5) *$M_m(f,\bar{c};\bar{c};S) = 0, \qquad$ for $m = 1,2,\ldots$*

6. AN ALGORITHM

An algorithm is proposed in this section for finding global minima of a discontinuous function under the assumptions (A1),(A2), (R), (M1), (M2) and (M3). Take a point $x_0 \in S$. If $c_0 =$

$f(x_0) = \bar{c} = \min_{x \in S} f(x)$, then x_0 is a global minimum point and c_0 is the global minimum value. The algorithm stops. In general, $c_0 > \bar{c}$, and $\mu(S \cap H_c) > 0$ by Lemma 5.1. Let

$$c_1 = M(f, c_0; S), \tag{6.1}$$

then

$$\bar{c} \le c_1 \le c_0. \tag{6.2}$$

In general, let

$$c_{k+1} = M(f, c_k; S), \qquad k = 0, 1, 2, \ldots. \tag{6.3}$$

If there is a positive integer k_0 such that

$$c_{k_0} = M(f, c_{k_0}; S), \tag{6.4}$$

then the algorithm is terminated: $\bar{c} = c_{k_0}$ and $\bar{H} = S \cap H_{c_{k_0}}$; otherwise we obtain a decreasing sequence

$$c_0 \ge c_1 \ge \cdots \ge c_k \ge c_{k+1} \ge \cdots \ge \bar{c} \tag{6.5}$$

and a monotone sequence of sets

$$S \cap H_{c_0} \supset S \cap H_{c_1} \supset \cdots \supset S \cap H_{c_k} \supset S \cap H_{c_{k+1}} \supset \ldots. \tag{6.6}$$

The limits exist. Let

$$\tilde{c} = \lim_{k \to \infty} c_k \qquad \text{and} \qquad \bar{H} = \lim_{k \to \infty} H_k$$

THEOREM 6.1. *Under the assumption (A1), (A2), (R), (M1), (M2) and (M3), $\tilde{c} = \bar{c}$ is the global minimum value and $\bar{H}$ is the set of global minimisers of f over S.*

PROOF. If the algorithm is terminated at a finite number of steps, then we have (6.4) and $\bar{c} = c_{k_0}$. If the algorithm does not stop, we obtain from (6.3), by letting $k \to \infty$,

$$\tilde{c} = M(f, \tilde{c}; S) \tag{6.7}$$

Hence, by Theorem 5.1, $\tilde{c} = \bar{c}$ is the global minimum value. Now, let $x \in \bar{H} \cap S$, then for each k (or $k > k_0$) we have $f(x) \le c_k$. Letting $k \to \infty$ (or setting $k = k_0$), we obtain

$$f(x) \le \bar{c} \tag{6.8}$$

But $f(x) \ge \bar{c}$ for all $x \in S$. Hence, $\bar{H} = \{x | f(x) = \bar{c}; x \in S\}$, i.e., $\bar{H}$ is the set of global minimisers. ∎

In applications we can use modified variance condition to verify if $V_1 = V_1(f, c_k; S) < \varepsilon$, where $\varepsilon > 0$ is the precision given in advance. If $V_1 \ge \varepsilon$, then the procedure is not terminated.

Note that the errors at each step in the algorithm will not be accumulated. Suppose we calculate $c_1 = M(f, c_0; S)$ with an error Δ_1 and obtain $d_1 = c_1 + \Delta_1$, then calculate $c_2 = M(f, d_1; S)$ with an error Δ_2 and obtain $d_2 = c_2 + \Delta_2$, and so on. In general, we have

$$c_k = M(f, d_{k-1}; S) \qquad \text{and} \qquad \Delta_k = d_k - c_k, \quad k = 1, 2, \ldots \tag{6.9}$$

and obtain a decreasing sequence $\{d_k\}$. Let

$$\bar{d} = \lim_{k \to \infty} d_k \tag{6.10}$$

THEOREM 6.2. *Under the assumption of Theorem 6.1, $\bar{d}$ is the global minimum value if and only if*

$$\lim_{k \to \infty} \Delta_k = 0 \tag{6.11}$$

The following theorem shows that the algorithm has a descent property.

THEOREM 6.3. *Under the assumptions of Theorem 6.1, if there is a positive integer k_0 such that $c_{k_0} = M(f, c_{k_0}; S)$, or $S \cap H_{c_{k_0+1}} = S \cap H_{c_{k_0}}$ then the function f is constant on $S \cap H_{c_{k_0}}$.*

The algorithm can be implemented by the Monte Carlo technique like in [4] and [5].

Numerical tests show that the discontinuity of the objective function does not influence the computation procedure essentially. An industrial application (see [6]) shows that the algorithm is effective.

REFERENCES

1. Q. Zheng, A class of discontinuous functions and global optimization problems, in Chinese, *Numerical Mathematics, A Journal of Chinese Universities* **1**, 31–43 (1985).
2. Q. Zheng, Global optimization of a class of discontinuous functions, in Chinese, *Journal of Applied Sciences* **1**, 93–94 (1986).
3. Q. Zheng, Optimality conditions for global optimization (I) and (II), *Acta Mathematicae Applicatae Sinica* **1**, 66–78, 118–132 (1986).
4. Q. Zheng, B. Jiang and S. Zhuang, A method for finding a global extremum, in Chinese, *Acta Mathematicae Applicatae Sinica* **2** (1978).
5. Q. Zheng, Theory and methods for global optimization, Advances in Optimization and Control, Lecture Notes, 302, Springer-Verlag, 15–37 (1988).
6. X. Pan, S. Wang, H. Liu and Q. Zheng, The optimum design of the arrangement of needles on needling board of preneedling machine, *J. of Chinese Textile University* **4** (1986).
7. R.T. Rockafeller, *Convex Analysis*, Princeton University Press, Princeton N.J., (1970).
8. E.A. Galperin and Q. Zheng, Integral global optimization in functional spaces with application to optimal control, to appear.